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Burgarth, Daniel; Giovannetti, Vittorio

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A Protocol For Cooling and Controlling Composite Systems by Local Interactions

Daniel Burgarth[1] and Vittorio Giovannetti[2]
[1]*Computer Science Department, ETH Zürich, CH-8092 Zürich, Switzerland*
[2]*NEST-CNR-INFM & Scuola Normale Superiore, piazza dei Cavalieri 7, I-56126 Pisa, Italy*

We discuss an explicit protocol which allows one to externally cool and control a composite system by operating on a small subset of it. The scheme permits to transfer *arbitrary and unknown* quantum states from a memory on the network ("upload access") as well as the inverse ("download access"). In particular it yields a method for cooling the system.

I. INTRODUCTION

Repetitive applications of the same quantum transformation have been exploited to achieve noise protection [1], cooling, state preparation [2, 3, 4], and quantum state transfer [5]. Motivated by the above results, in Ref. [6] we developed a scheme for controlling larger systems when control is only assumed to be available on a subsystem. Once this is achieved, apart from cooling and state preparation, it is also possible to perform arbitrary quantum data processing (e.g. measurements, unitary rotations). This is similar in spirit to universal quantum interfaces of Ref. [7], but our approach allows us to specify explicit protocols and to give lower bounds for fidelities. These techniques are also related with the "asymptotic completeness" property introduced by Kümmerer and Maassen [4, 8] which allows one to control a system by coupling it with quantum mediators.

In the present paper we review the scheme of Ref. [6] by showing how arbitrary quantum states can be written into (i.e. prepared on) a large system, and read from it, by *local* control only. This implies that arbitrary quantum operations on the system state can be performed. An important specific task is the cooling of the system to its ground state. Using some heuristic argument, we will provide an estimate of the convergence time of the cooling and we will test it with some numerical examples. We develop the protocol in several steps. First, we show that the system of interest can be actively brought to its ground state by replacing its controlled part with fresh "cold" qubits from a memory. We then find that cooling implies that the information about the initial system state is transferred into the memory, and design a linear map that decodes this information. Since this map is generally not unitary, we use the polar decomposition to find its best unitary approximation. The fidelity of information decoding can then be lower bounded by the overlap of the system state with its ground state. Finally we design the reverse operation allowing us to transfer information from the memory to the system.

The material is organized as follows. In Sec. II the protocol is presented in its general lines. In Sec. III we give a detailed derivation of the coding and decoding transformations and derive bounds for the fidelities. Numerical estimations of the protocol performances are given in Sec. IV focusing on the case of locally controlled Heisenberg-like coupled spin networks. Conclusion and remarks are in Sec. V while technical material is presented in the Appendices.

II. THE PROTOCOL

Consider a composed system described by the Hilbert space $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_{\bar{C}} \otimes \mathcal{H}_M$. We assume that full control (the ability to prepare states and apply unitary transformations) is possible on system C and M, but no control is available on system $\bar{C}$. Moreover, we assume that C and $\bar{C}$ are coupled by a time-independent Hamiltonian H. We show here that under certain assumptions, if the system $C\bar{C}$ is initialized in some arbitrary state we can transfer ("download") this state into the system M by applying some operations between M and C only. Likewise, by initializing the system M in the correct state, we can transfer ("upload") arbitrary states on the system $C\bar{C}$. The system M functions as a *quantum memory* and must be at least as large as the system $C\bar{C}$. As sketched in Fig. 1 we can imagine the memory to be split into sectors M_ℓ, having the same dimension of C, i.e. $\mathcal{H}_M = \bigotimes_{\ell=1}^{L} \mathcal{H}_{M_\ell}$ with $\dim\mathcal{H}_{M_\ell} = \dim\mathcal{H}_C$.

A. Downloading info from $C\bar{C}$ to M

The downloading protocol we present here is composed by two stages: a *swapping* stage, in which at regular time intervals we couple the subsystem C to the first L memories M; and a *decoding* stage in which we apply a unitary transformation to the first L memories in order to recover the initial state of $C\bar{C}$. As we will see for any finite L our analysis does not guarantee that the fidelity between the recovered state and the initial state of $C\bar{C}$ is perfect. However, in Sec. III A it will be shown that by augmenting L one can make the fidelity arbitrarily close to one.

We assume that the memory is initialized in a factorized state of the form

$$|0\rangle_M \equiv \bigotimes_{\ell=1}^{L} |0\rangle_{M_\ell} \ , \tag{1}$$

where $|0\rangle$ is a state whose properties will be specified in the following. To download a generic state, we let the

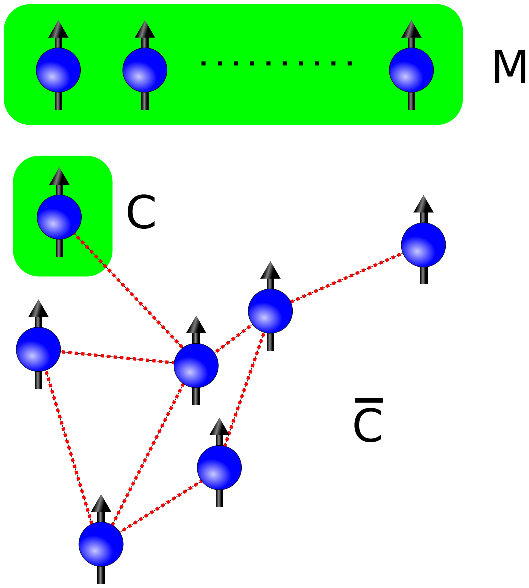

FIG. 1: Example of the model discussed in the text. Here the system $C\bar{C}$ is formed by 7 spins characterized by some time-independent Hamiltonian H (the coupling are represented by red lines connecting the spins). The system $C\bar{C}$ can only be controlled by acting on a (small) subsystem C (in this case represented by the uppermost spin of the network). The coupling H can - in some cases - *mediate* the local control on C to the full system $C\bar{C}$. In our case, system C is controlled by performing regular swap operations S_ℓ between it and a 2-dim quantum memory M_ℓ.

system $C\bar{C}$ to evolve for a while according to its Hamiltonian H, perform a unitary gate which couples C to one of the sectors of M, let $C\bar{C}$ evolve again and so forth. More specifically, at step ℓ of the protocol we perform a unitary swap $S_\ell \equiv S_{CM_\ell}$ between system C and system M_ℓ [9]. After the Lth swap operation the protocol stops. This is the *swapping* stage and it is characterized by the unitary operator

$$W \equiv S_L U S_{L-1} U \cdots S_\ell U \cdots S_1 U \;, \tag{2}$$

where $U = \exp[-iHt]$ is the time-evolution of $C\bar{C}$ for some fixed time interval t. As discussed in Ref. [6], the reduced evolution of the system $\bar{C}$ under the transformation (2) can be expressed in terms of the following completely positive CP map [10]

$$\tau(\rho_{\bar{C}}) \equiv \mathrm{Tr}_C \left[U \left(\rho_{\bar{C}} \otimes |0\rangle_C\langle 0| \right) U^\dagger \right] \;, \tag{3}$$

where $|0\rangle_C$ is the state that is swapped in from the memory and $\mathrm{Tr}_C[\cdots]$ indicate the partial trace over the subsystem C. Indeed, after L swaps the state of $\bar{C}$ is obtained by taking the partial trace with respect to C and M of the vector $W\left(|\psi\rangle_{C\bar{C}}|0\rangle_M\right)$ where $|\psi\rangle_{C\bar{C}}$ is the initial state of $C\bar{C}$, i.e.

$$\begin{aligned} \rho_{\bar{C}}^{(L)} &= \mathrm{Tr}_{CM}\left[W \left(|\Psi\rangle_{C\bar{C}}\langle\Psi| \otimes |0\rangle_M\langle 0| \right) W^\dagger \right] \\ &= \underbrace{\tau\circ\tau\circ\cdots\circ\tau}_{L-1 \text{ times}}(\rho'_{\bar{C}}) \equiv \tau^{L-1}(\rho'_{\bar{C}}) \;, \end{aligned} \tag{4}$$

where "$\circ$" represents the composition of super-operators [10] and

$$\rho'_{\bar{C}} \equiv \mathrm{Tr}_C \left[U|\psi\rangle_{C\bar{C}}\langle\psi| U^\dagger \right] \;, \tag{5}$$

(for an explicit derivation of this expression see Appendix A).

Our main assumption is that the map τ is ergodic with pure fixed point which we denote as $|0\rangle_{\bar{C}}$. Explicitly this means that the only state which is left invariant by τ is the vector $|0\rangle_{\bar{C}}$, i.e.

$$\tau(\rho_{\bar{C}}) = \rho_{\bar{C}} \quad \Longleftrightarrow \quad \rho_{\bar{C}} = |0\rangle_{\bar{C}}\langle 0| \;. \tag{6}$$

As shown in Refs. [11, 12] this implies that the channel τ is relaxing (mixing), that is

$$\lim_{n\to\infty} \tau^n(\sigma_{\bar{C}}) = |0\rangle_{\bar{C}}\langle 0| \;, \tag{7}$$

for *all* $\sigma_{\bar{C}}$. This condition gives rise to the controllability of the system. Indeed from Eq. (7) it follows that for sufficiently large L, an initial state of the form $|\psi\rangle_{C\bar{C}} \otimes |0\rangle_M$ can be approximated as

$$W\left(|\psi\rangle_{C\bar{C}} \otimes |0\rangle_M\right) \approx |0\rangle_{C\bar{C}} \otimes |\Phi(\psi)\rangle_M . \tag{8}$$

The right hand side of this equation factorizes into pure states because the transformation W is unitary, and because both the initial state of $C\bar{C}$ and M and the final state of $C\bar{C}$ are pure. This implies that, in the asymptotic limit of infinitely many protocol steps (i.e. $L \gg 1$), the system $C\bar{C}$ has been "cooled" into the state $|0\rangle_{C\bar{C}}$ while all the information regarding the initial state $|\psi\rangle_{C\bar{C}}$ must be contained in the vector $|\Phi(\psi)\rangle_M$ [13]. Furthermore, it is at least intuitively clear that such information can be recovered by the application of a proper unitary "decoding" operation $V^\dagger$ on M which does not depend on the input state of the system (*decoding* stage), i.e. [14]

$$V^\dagger \, |\Phi(\psi)\rangle_M \approx |\psi\rangle_M \;. \tag{9}$$

At a mathematical level, the convergence of the downloading protocol described above only depends upon the invariant property (6) — see Ref. [6]. In Sec. III we will briefly review such a proof and provide a characterization of the unitary transformation V.

B. Uploading info from M to $C\bar{C}$

For uploading states on the system $C\bar{C}$, we again make use of the unitarity of W. Let us again first give a simple hand-waving argument why this is possible.

Suppose you want to drive the system into the state $|\psi\rangle_{C\bar{C}}$. To do this, you first use the downloading protocol to make sure that the system is in the state $|0\rangle_{C\bar{C}}$ ("cooling"). Then you bring the memories into the state $|\Phi(\psi)\rangle_M$ they would have been ended up in case one was trying to download $|\psi\rangle_{C\bar{C}}$ from $C\bar{C}$ into M as in Eq. (8). Now the quantum recurrence theorem [16] implies that there is a m such that

$$\begin{aligned} |\psi\rangle_{C\bar{C}} \otimes |0\rangle_M \;&\approx\; W^m \big(|\psi\rangle_{C\bar{C}} \otimes |0\rangle_M\big) \\ &\approx\; W^{m-1} \big(|0\rangle_{C\bar{C}} \otimes |\Phi(\psi)\rangle_M\big) \;, \end{aligned} \tag{10}$$

where we have made use of Eq. (8). Hence by applying W m times you have approximately initialized $|\psi\rangle_{C\bar{C}}$. Of course it remains to be shown that *unknown* states can be written to the system, too. This and the mathematical details will be discussed in the next section. Another problem with Eq. (10) is that the recurrence parameter m typically needs to be *huge*, scaling double exponentially with the number of qubits in the system. There are however alternative, more efficient ways of implementing an uploading process from M to $C\bar{C}$. The simplest one is of course to apply the inverse transformation $W^{-1} = W^\dagger$ to the state of Eq. (8). Indeed the protocol we presented in Ref. [6] is based on this idea, which is a generalization of [4]. Unfortunately the inverse of W is generally unphysical in the sense that it requires backward time evolutions U^{-1}, i.e. one would have to wait *negative* time steps between the swaps (see however Ref. [15] for cases in which such an inverse time evolution can be implemented by clever external control techniques).

To overcome this problem we introduce an extra hypothesis. Specifically we consider the case in which the invariant property (6) holds also for the channel τ' obtained from Eq. (3) by replacing U with $U^\dagger$, i.e.

$$\tau'(\rho_{\bar{C}}) \equiv \mathrm{Tr}_C \left[U^\dagger \left(\rho_{\bar{C}} \otimes |0\rangle_C\langle 0|\right) U \right] \;. \tag{11}$$

Under this condition, similarly to the case of W discussed in the previous section, one can verify that in the limit of large L, *i)* the transformation

$$W' \equiv S_L U^\dagger S_{L-1} U^\dagger \cdots S_\ell U^\dagger \cdots S_1 U^\dagger \;, \tag{12}$$

applied to $|\psi\rangle_{C\bar{C}} \otimes |0\rangle_M$ will converge to a vector of the form $|0\rangle_{C\bar{C}} \otimes |\Phi'(\psi)\rangle_M$; *ii)* there exists a unitary transformation V' which does not depend upon $|\psi\rangle$ and which applied to M gives

$$V'^\dagger |\Phi'(\psi)\rangle_M \approx |\psi\rangle_M \;. \tag{13}$$

From this we can write

$$\begin{aligned} &|\psi\rangle_{C\bar{C}} \otimes |0\rangle_M \approx (W')^\dagger\, V' \left(|0\rangle_{C\bar{C}} \otimes |\psi\rangle_M\right) \\ &\approx (U S_1 \cdots U S_\ell \cdots U S_{L-1} U S_L)\, V' \left(|0\rangle_{C\bar{C}} \otimes |\psi\rangle_M\right). \end{aligned} \tag{14}$$

What it is relevant for us is the fact that now the unitary transformation on the input state $|0\rangle_{C\bar{C}} \otimes |\psi\rangle_M$ does not involve "time-reversal" evolutions U^{-1} but only "proper" time evolution U. Therefore, by imposing the condition (6) on τ', we are able to define an uploading protocol which transfers an unknown state $|\psi\rangle$ from M to $C\bar{C}$. Similarly to the downloading scheme it is composed by two stages: an *encoding* stage in which we apply the unitary transformation V' to "prepare" the memory M and a *swapping* stage in which we apply the unitary

$$(W')^\dagger = U S_1 \cdots U S_\ell \cdots U S_{L-1} U S_L \;, \tag{15}$$

by recursively coupling C to the M through swaps.

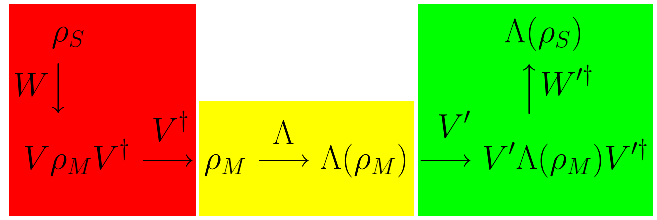

FIG. 2: Summary of the scheme: any CP map Λ can be applied to the system by acting on the memory instead through the transformations shown in the figure. The red and green areas represent the downloading and uploading part of the protocol, respectively. The unitary operators W and $W'^\dagger$ of Eqs. (2) and (15) are generated by acting on the memory and a small subsystem of the system only; $V^\dagger$ and V' are instead the decoding and encoding unitary transformations introduced in Eqs. (9) and (13), respectively — see also Sec. III.

Two remarks are mandatory. On one hand, as in the case of the downloading protocol, the convergence of the transformation (14) only depends upon the invariant condition (6) of the channel τ'. On the other hand, there exists a large class of physically relevant Hamiltonians H (e.g. nearest neighbors Heisenberg coupling Hamiltonians) for which both τ and τ' verify the such condition — we refer the reader to Ref. [6] for details. For such Hamiltonians, our analysis will yield both a simple downloading and uploading mechanism. Putting these two elements together one can also realize more sophisticated controls. For instance, as shown in Fig. 2, one can perform any quantum transformation Λ on $C\bar{C}$ by first downloading its state on M, transforming it, and finally uploading the final state back into the system.

III. CODING TRANSFORMATION

In this section we derive the decoding transformation $V^\dagger$ that relates states on the memories M to the states on $C\bar{C}$ in the downloading protocol. To do so we exploit the formal decomposition of the evolved state of the system after L steps (see Appendix B). The encoding transformation V' of the uploading protocol can be obtained in a similar way.

Consider an orthonormal basis $\{|\psi_k\rangle_{C\bar{C}}\}$ of $\mathcal{H}_{C\bar{C}}$. According to Eq. (B1) after L swaps it becomes

$$W\big(|\psi_k\rangle_{C\bar{C}}|0\rangle_M\big) \qquad (16)$$
$$= |0\rangle_C \otimes \left[\sqrt{\eta_k}|0\rangle_{\bar{C}}|\phi_k\rangle_M + \sqrt{1-\eta_k}|\Delta_k\rangle_{\bar{C}M}\right] ,$$

where $|\Delta_k\rangle_{\bar{C}M}$ is a vector orthogonal to $|0\rangle_{\bar{C}}$ and $\eta_k \approx 1$ as in Eq. (B6). This equation shows that with high probability, the transformation W maps the orthonormal vectors $|\psi_k\rangle_{C\bar{C}}$ into the vectors $|\phi_k\rangle_M$ of the first L memories. For any finite choice of L, the latter are typically not mutually orthogonal. However one can use Eq. (B6) to show that in the limit of large L the vectors $|\phi_k\rangle_M$ become approximately orthogonal. Indeed from the unitarity of W and from Eq. (16) and (B3) we can establish the following identity

$$\begin{aligned}
&{}_M\langle\phi_k|\phi_{k'}\rangle_M \qquad (17)\\
&= \sqrt{\eta_k\,\eta_{k'}}\;\delta_{kk'} + \sqrt{\eta_k\,(1-\eta_{k'})}\;{}_{\bar{C}CM}\langle\psi_k 0|\tilde{\Delta}_{k'}\rangle_{\bar{C}CM}\\
&+\sqrt{\eta_{k'}\,(1-\eta_k)}\;{}_{\bar{C}CM}\langle\tilde{\Delta}_k|\psi_{k'}0\rangle_{\bar{C}CM}\\
&+\sqrt{(1-\tilde{\eta}_k)(1-\tilde{\eta}_{k'})}\;{}_{C\bar{C}M}\langle\tilde{\Delta}_k|\tilde{\Delta}_{k'}\rangle_{C\bar{C}M} \; .
\end{aligned}$$

To simplify this expression we define $\eta_0 \equiv \min_k \eta_k$. Since Eq. (B6) applies to all η_k the parameter η_0 must satisfy the inequality

$$1-\eta_0 \leqslant K\;(L-1)^{d_{\bar{C}}}\;\kappa^{L-1} \; . \qquad (18)$$

Furthermore from Eq. (17) it follows that for $k \neq k'$ one has

$$\begin{aligned}
&|{}_M\langle\phi_k|\phi_{k'}\rangle_M|\\
&\leqslant |\sqrt{\eta_k\,(1-\eta_{k'})}\;|{}_{\bar{C}CM}\langle\psi_k 0|\tilde{\Delta}_{k'}\rangle_{\bar{C}CM}|\\
&+\sqrt{\eta_{k'}\,(1-\eta_k)}\;|{}_{\bar{C}CM}\langle\tilde{\Delta}_k|\psi_{k'}0\rangle_{\bar{C}CM}|\\
&+\sqrt{(1-\tilde{\eta}_k)(1-\tilde{\eta}_{k'})}\;|{}_{C\bar{C}M}\langle\tilde{\Delta}_k|\tilde{\Delta}_{k'}\rangle_{C\bar{C}M}|\\
&\leqslant 2\sqrt{1-\eta_0}+(1-\eta_0)\;\leqslant\;3\sqrt{1-\eta_0}\,, \qquad (19)
\end{aligned}$$

which according to Eq. (18) and using the fact that the parameter κ is strictly smaller than 1, shows that for large L the vectors $|\phi_k\rangle_M$ and $|\phi_{k'}\rangle_M$ become orthogonal.

Define then the linear operator D on $\mathcal{H}_M$ which performs the following transformation

$$D|\psi_k\rangle_M = |\phi_k\rangle_M \; , \qquad (20)$$

with $|\psi_k\rangle_M$ being orthonormal vectors of M which represent the states $\{|\psi_k\rangle_{C\bar{C}}\}$ of $\mathcal{H}_{C\bar{C}}$. Formally they are obtained by a partial isometry from $\bar{C}C$ to M and are "good" representations of the $|\psi_k\rangle_{C\bar{C}}$. The operator D in some sense "corrects" the non-orthogonality of the $|\phi_k\rangle_M$: indeed its inverse (when definable) allows us to pass from these approximate images of the $|\psi_k\rangle_{C\bar{C}}$ to the good representations $|\psi_k\rangle_M$. Therefore D^{-1} seems to be a good candidate for defining our decoding transformation V. Unfortunately however D is NOT unitary (it maps an orthonormal set of states into a non-orthonormal one) and typically will not be even invertible.

The idea is then to replace D with its *best unitary approximation* V [17, p 432]. The latter is obtained by taking the polar decomposition of D, i.e.

$$D = PV \; , \qquad (21)$$

with P positive semidefinite. The unitary V minimizes the norm distance from D yielding the inequality

$$\begin{aligned}
||D-V||_2 \;&=\; \sqrt{\sum_k \left[\sqrt{\lambda_k}-1\right]^2} \leqslant \sqrt{\sum_k |\lambda_k - 1|}\\
&\leqslant\; \sqrt{3}\;d_{C\bar{C}}\;(1-\eta_0)^{1/4}\;, \qquad (22)
\end{aligned}$$

where we introduced the eigenvalues λ_k of $D^\dagger D$ and used Eq. (19) to bound them according to the inequality $|\lambda_k - 1| \leqslant 3\;d_{C\bar{C}}\;\sqrt{1-\eta_0}$ (in these expressions $d_{C\bar{C}} = d_C d_{\bar{C}}$ is the dimension of the system $C\bar{C}$ and $\|\Theta\|_2$ stands for $\sqrt{\sum_{kk'}|\Theta_{kk'}|^2}$ with $\Theta_{kk'}$ being the matrix elements of the operator Θ). The inequalities (22) and (18) show that for $L \to \infty$, D can be approximated arbitrarily well by the unitary V. We can hence define $V^\dagger$ as our decoding transformation which inverts the mapping (20) and transforms the "bad" representations $|\phi_k\rangle_M$ of the $|\psi_k\rangle_{C\bar{C}}$ into the "good" representations $|\psi_k\rangle_M$. It is worth stressing that, by construction, V does not depend upon the input state $|\psi\rangle_{C\bar{C}}$ of the system $C\bar{C}$.

As mentioned in the introduction of this section, a similar procedure can used to defined the encoding protocol of the uploading protocol. Without entering into the details we simply notice that in this case D and the vectors $|\phi_k\rangle_M$ will be defined by replacing W of Eq. (16) with the transformation W' of Eq. (12). Taking the polar decomposition of such new D it will yield the unitary V' which will be used as encoding for the uploading scheme.

In the following section we will evaluate the transfer fidelities associated with such a choice of decoding and encoding transformation, showing that they can arbitrarily increased by choosing L sufficiently high.

A. Fidelity of the downloading protocol

Let $|\psi\rangle_{C\bar{C}} = \sum_k \alpha_k|\psi_k\rangle_{C\bar{C}}$ be a generic input state of $C\bar{C}$. To evaluate the downloading fidelity $F_{\rm down}$ associated with our decoding scheme we need to compare the state of M at the end of the protocol with the state $|\psi\rangle_M = \sum_k \alpha_k|\psi_k\rangle_M$, i.e.

$$F_{\rm down}(\psi) \;\equiv\; {}_M\langle\psi|V^\dagger\;R_M\;V|\psi\rangle_M \; . \qquad (23)$$

Here $V^\dagger$ is the decoding transformation defined in the previous section, and R_M is the state of the memory after the application of the unitary W, i.e.

$$\begin{aligned}
R_M \;&\equiv\; {\rm Tr}_{C\bar{C}}\left[W(|\psi\rangle_{C\bar{C}}\langle\psi|\otimes|0\rangle_M\langle 0|)W^\dagger\right]\\
&=\; \eta\;|\phi\rangle_M\langle\phi| + (1-\eta)\;\sigma_M \; . \qquad (24)
\end{aligned}$$

In the above expression we used Eqs. (B1) and (B2) and introduced the density matrix $\sigma_M \equiv \mathrm{Tr}_{\bar{C}}[|\Delta\rangle_{\bar{C}M}\langle\Delta|]$. By linearity we get

$$F_{\rm down}(\psi) = \eta\, |_M\langle\phi|V|\psi\rangle_M|^2 + (1-\eta)\; _M\langle\psi|V^\dagger\,\sigma_M\,V|\psi\rangle_M \geqslant \eta\, |_M\langle\phi|V|\psi\rangle_M|^2 \,. \quad (25)$$

We now bound the term on the right hand side as follows

$$|_M\langle\phi|V|\psi\rangle_M| = |_M\langle\phi|V - D + D|\psi\rangle_M| \geqslant |_M\langle\phi|D|\psi\rangle_M| - |_M\langle\phi|D - V|\psi\rangle_M| \,, \quad (26)$$

and use the inequality (22) to write

$$|_M\langle\phi|D-V|\psi\rangle_M| \leqslant ||D-V||_2 \leqslant \sqrt{3}\; d_{C\bar{C}}\; (1-\eta_0)^{1/4}\,.$$

If $|\psi\rangle_M$ was a basis state $|\psi_k\rangle_M$, then $|_M\langle\phi|D|\psi\rangle_M| = 1$ by the definition Eq. (20) of D. For generic $|\psi\rangle_M$ instead we can use the linearity to find after some algebra that

$$\sqrt{\eta}\; |_M\langle\phi|D|\psi\rangle_M| \;\geqslant \sqrt{\eta_0}\; - \;3\; d_{C\bar{C}}\; \sqrt{1-\eta_0}\,. \quad (27)$$

Therefore Eq. (26) gives

$$\sqrt{\eta}\; |_M\langle\phi|V|\psi\rangle_M| \;>\; \sqrt{\eta_0}\; - 5\; d_{C\bar{C}}\; (1-\eta_0)^{1/4}\,, \quad (28)$$

which replaced in Eq. (25) yields

$$F_{\rm down}(\psi) \;\geqslant\; \eta_0\; - 10\; d_{C\bar{C}}\; (1-\eta_0)^{1/4}\,, \quad (29)$$

for all input states $|\psi\rangle_{C\bar{C}}$. According to Eq. (18) it then follows that by choosing L sufficiently big our downloading protocol will yield transferring fidelities arbitrarily close to one.

B. Fidelity of the uploading protocol

Following the analysis of Sec. (II B) the fidelity for uploading a state $|\psi\rangle_M$ into $\bar{C}C$ is given by

$$F_{\rm up}(\psi) \equiv \; _{C\bar{C}}\langle\psi|\mathrm{Tr}_M\left[W'^\dagger V'\left(|\psi\rangle_M\langle\psi| \otimes |0\rangle_{\bar{C}C}\langle 0|\right) V'^\dagger W'\right]|\psi\rangle_{C\bar{C}}. \quad (30)$$

A lower bound for this quantity is obtained by replacing the trace over M with the expectation value on $|0\rangle_M$, i.e.

$$\begin{aligned} F_{\rm up}(\psi) &\geqslant \; _{C\bar{C}}\langle\psi|_M\langle 0|W'^\dagger V'\big(|\psi\rangle_M\langle\psi| \otimes |0\rangle_{\bar{C}C}\langle 0|\big) \times V'^\dagger W'|0\rangle_M|\psi\rangle_{C\bar{C}} \\ &= \left|_{C\bar{C}}\langle 0|_M\langle\psi|V'^\dagger W'|0\rangle_M|\psi\rangle_{C\bar{C}}\right|^2 \\ &= \eta'\; \left|_M\langle\psi|V'^\dagger|\phi\rangle_M\right|^2 = \eta'\; |_M\langle\phi|V'|\psi\rangle_M|^2\,. \end{aligned} \quad (31)$$

In deriving this equation we used Eq. (15) and a decomposition of the form of Eq. (B1) to simplify the vector $W'|0\rangle_M|\psi\rangle_{C\bar{C}}$. In this case η' is defined as in Eq. (B5) with τ being replaced by τ' of Eq. (11). Since we are assuming that this CP map satisfies the condition (6) it follows that also η' obeys an inequality of the form (B6) with K and κ replaced by new constants K' and $\kappa' \in]0,1[$. We also notice that last term of Eq. (31) has the same form of the lower bound (25) of the downloading fidelity. Therefore, by applying the same derivation of the previous section we can write

$$F_{\rm up}(\psi) \geqslant \eta_0'\; - 10\; d_{C\bar{C}}\; (1-\eta_0')^{1/4}\,, \quad (32)$$

with

$$1 - \eta_0' \leqslant K'\; (L-1)^{d_{\bar{C}}}\; (\kappa')^{L-1}. \quad (33)$$

This shows that, as in the downloading case, also the uploading fidelity converges to unity in the limit of large L.

IV. EFFICIENCY OF COOLING

In this section we provide some numerical estimation of the quantities η of Eq. (B5) which measure the probability of finding the state $\bar{C}$ in $|0\rangle_{\bar{C}}$. As seen in the previous sections this is the fundamental parameter to bound the fidelities of both the downloading and uploading protocol. Moreover, given an initial state $|\psi\rangle_{C\bar{C}}$, η measures the success probability of "cooling" it down to the state $|0\rangle_{C\bar{C}}$ during the downloading process. According to Eq. (B6) the quantity η will *asymptotically* converge exponentially fast to unity. However Eq. (B6) does not tell us from which point onwards the convergence is exponentially fast, so it would be nice to have alternative ways to estimate the convergence speed.

To simplify the analysis in the following, we will concentrate on the spin network model of Fig. 1 assuming a Heisenberg Hamiltonian of the form

$$H = \sum_{(j,j')\in G} d_{j,j'}\big(X_j X_{j'} + Y_j Y_{j'} + Z_j Z_{j'}\big)\,, \quad (34)$$

which conserves the total magnetization along the z axis (here X_j, Y_j and Z_j are the Pauli operators of the j-th spin and the summation is performed over all the edges of the weighted graph G associated with the network). Moreover we will take the vector $|0\rangle_C$ to be the configuration where all the qubits of C are in the spin-down state, i.e. $|0\rangle_C \equiv |00\cdots 00\rangle_C$. For this choice of the controller state our main assumption of ergodicity Eq. (6) is numerically found to be correct for the coupling graph depicted in Fig. 1. The fixed point is given by $|0\rangle_{\bar{C}} \equiv |00\cdots 00\rangle_{\bar{C}}$ (more general conditions of ergodicity for Heisenberg models are given in [6]). In this context η coincides then with the probability $P_0^{(L)}$ of finding no excitations on the system after L steps of the protocol. Some numerical examples showing the dependence of η upon the initial state are presented in Fig. 3. As expected, asymptotically $P_0^{(L)}$ is seen to converge exponentially fast.

An approximate estimation of $P_0^{(L)}$ can be easily obtained by looking at the *average* number of spin-up on

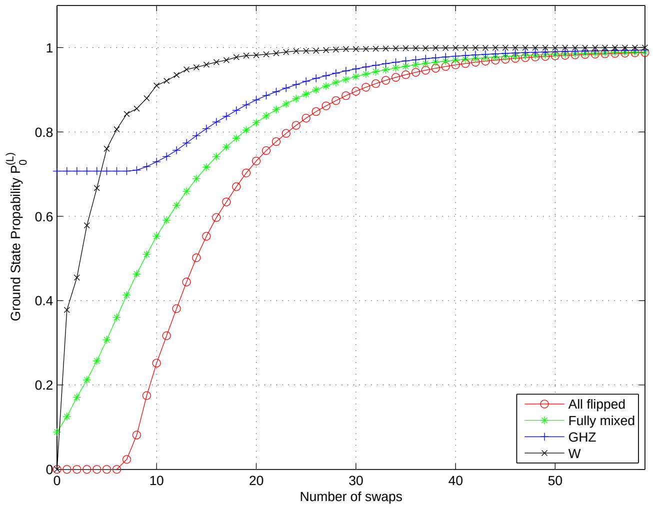

FIG. 3: Convergence of the cooling protocol for the weighted graph of Fig. 1 where the couplings among the spins is given by the Hamiltonian (34) (the values of constants $d_{j,j'}$ have been chosen to be proportional to the length of the graph edge). Four different states $|\psi\rangle_{C\bar{C}}$ are considered: the fully flipped state $|1111111\rangle_{C\bar{C}}$, a GHZ state $(|0000000\rangle_{C\bar{C}} + |1111111\rangle_{C\bar{C}})/\sqrt{2}$, a fully mixed state $\rho_{C\bar{C}} = \mathbb{1}_{C\bar{C}}/2^7$, and a W state $\frac{1}{\sqrt{7}}(|1000000\rangle_{C\bar{C}} + |0100000\rangle_{C\bar{C}} + \cdots + |0000001\rangle_{C\bar{C}})$.

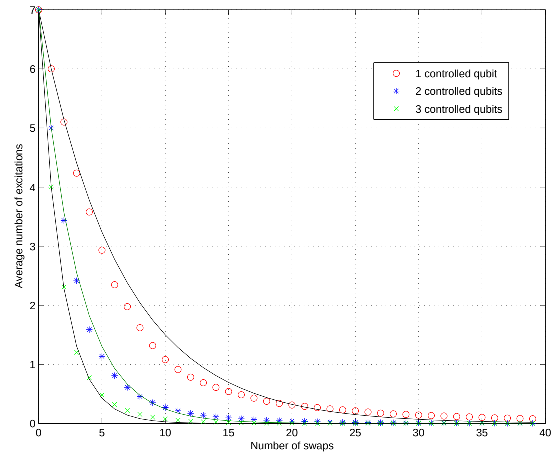

FIG. 4: Comparison of the approximation Eq. (37) with exact numerical results. Shown is the average number of excitations $\left\langle \hat{N} \right\rangle_{C\bar{C}}$ on an open Heisenberg spin chain with 7 sites and equal couplings as a function of the number of swaps to the memory. The initial state is taken to be $|1111111\rangle_{C\bar{C}}$, i.e. with a maximal number of excitations. The three curves correspond to different sizes of the region $|C|$ controlled by Alice, and the time interval t has been chosen for each curve independently to fit the approximation given in Eq. (37).

$C\bar{C}$ after L swaps, i.e.

$$\left\langle \hat{N} \right\rangle^{(L)}_{C\bar{C}} \equiv \mathrm{Tr}_{C\bar{C}} \left[\hat{N} \, \rho^{(L)}_{C\bar{C}} \right] , \qquad (35)$$

with $\rho^{(L)}_{C\bar{C}} \equiv \mathrm{Tr}_M \left[W \left(|\psi\rangle_{C\bar{C}} \langle\psi| \otimes |0\rangle_M \langle 0| \right) W^\dagger \right]$ being the reduced density matrix of $C\bar{C}$ and with $\hat{N} \equiv \sum_{k \in C, \bar{C}} (Z_k + 1)/2$ (here Z_k is the z-Pauli matrix of the k-th spin). These quantities are related by

$$P_0^{(L)} \geqslant 1 - \left\langle \hat{N} \right\rangle^{(L)}_{C\bar{C}} . \qquad (36)$$

To get an approximation for the average number of excitations on the graph we assume now that the time interval t is chosen such that U shuffles the excitations on the graph in a fully random way. For specific systems and specific times intervals, this "classical" behavior might not be true due to interferences, but for general times it is a good approximation (see Fig. 4). Let $|C|$ be the number of edges on the graph controlled by Alice, and $|\bar{C}|$ the number of uncontrolled edges. On average, each swap takes approximately a ratio $|C|/(|C| + |\bar{C}|)$ of excitations from the graph to the memory. We then get

$$\left\langle \hat{N} \right\rangle^{(L)}_{C\bar{C}} \approx \left\langle \hat{N} \right\rangle^{(0)}_{C\bar{C}} \left(\frac{1}{1 + |C| / |\bar{C}|} \right)^L . \qquad (37)$$

This is a reasonable result which shows that the fidelity depends on the initial number of excitations and on the relative size of the controlled region with respect to the uncontrolled region.

V. CONCLUSION

We have given an explicit protocol for controlling and cooling a large permanently coupled system by accessing a small subsystem only. As we have shown, the applicability relies only on the invariant property (6) of a CPT map. Since we had to assume a large quantum memory in order to control the system, this protocol is not useful for replacing control in a homogeneous setup, but may

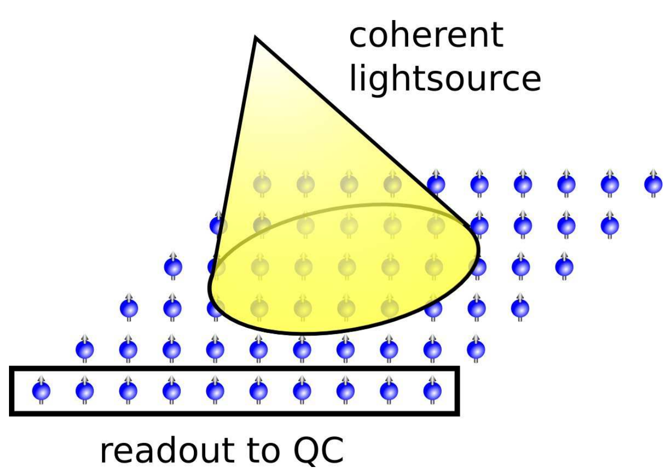

FIG. 5: A CCD-like application of our protocol could allow a light sensitive array of qubits to be read out *coherently* by a quantum computer without "disturbing" the qubits much.

well have applications in inhomogeneous scenarios (when control is harmful or expensive in some regions but easy in others). For example, we imagine a CCD-like application, in which a set of permanently coupled qubits is read out by a Quantum Computer in a coherent manner (see Fig. 5).

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APPENDIX A: EVOLUTION OF $\bar{C}$

Here we derive the evolution (4) of the subset $\bar{C}$ in terms of the CP map (3). First rewrite the reduced density matrix (4) as follows

$$\begin{aligned}\rho_{\bar{C}}^{(L)} &= \mathrm{Tr}_{CM}\left[W\left(|\Psi\rangle_{C\bar{C}}\langle\Psi|\otimes|0\rangle_M\langle 0|\right)W^\dagger\right]\\ &= \mathrm{Tr}_C\Big[\cdots\mathrm{Tr}_{M_2}\Big[S_2U\Big(\mathrm{Tr}_{M_1}\Big[S_1U\Big(|\Psi\rangle_{C\bar{C}}\langle\Psi|\otimes|0\rangle_{M_1}\langle 0|\Big)U^\dagger S_1^\dagger\Big]\otimes|0\rangle_{M_2}\langle 0|\Big)U^\dagger S_2^\dagger\Big]\cdots\Big]\,. \end{aligned} \tag{A1}$$

For the sake of clarity it is useful to explicitly denote the subsystems on which the various operators are acting on (e.g. Θ_{AB} indicates that the operator Θ acts non trivially only on the subsystems A and B, while it is the identity elsewhere). By doing so and by using the properties [9] of the swap it is easy to verify the following identities:

$$\begin{aligned}&\mathrm{Tr}_{M_\ell}\Big[S_\ell U\Big(\rho_{\bar{C}}\otimes|0\rangle_C\langle 0|\otimes|0\rangle_{M_\ell}\langle 0|\Big)U^\dagger S_\ell^\dagger\Big] = \mathrm{Tr}_{M_\ell}\Big[S_{CM_\ell}U_{C\bar{C}}\Big(\rho_{\bar{C}}\otimes|0\rangle_C\langle 0|\otimes|0\rangle_{M_\ell}\langle 0|\Big)U_{C\bar{C}}^\dagger S_{CM_\ell}^\dagger\Big]\\ &= \mathrm{Tr}_{M_\ell}\Big[S_{CM_\ell}U_{C\bar{C}}\Big(S_{CM_\ell}^\dagger S_{CM_\ell}\Big)\Big(\rho_{\bar{C}}\otimes|0\rangle_C\langle 0|\otimes|0\rangle_{M_\ell}\langle 0|\Big)\Big(S_{CM_\ell}^\dagger S_{CM_\ell}\Big)U_{C\bar{C}}^\dagger S_{CM_\ell}^\dagger\Big]\\ &= \mathrm{Tr}_{M_\ell}\Big[U_{M_\ell\bar{C}}\Big(\rho_{\bar{C}}\otimes|0\rangle_{M_\ell}\langle 0|\otimes|0\rangle_C\langle 0|\Big)U_{M_\ell\bar{C}}^\dagger\Big] = \tau(\rho_{\bar{C}})\otimes|0\rangle_C\langle 0|\,,\end{aligned} \tag{A2}$$

which holds for all $\rho_{\bar{C}}$ and ℓ. Equation (4) then follows by replacing this into Eq. (A1) for all $\ell > 2$ and by using the identity

$$\mathrm{Tr}_{M_1}\Big[S_1U\Big(|\Psi\rangle_{C\bar{C}}\langle\Psi|\otimes|0\rangle_{M_1}\langle 0|\Big)U^\dagger S_1^\dagger\Big] = \mathrm{Tr}_{M_1}\Big[S_{CM_1}U_{C\bar{C}}\Big(|\Psi\rangle_{C\bar{C}}\langle\Psi|\otimes|0\rangle_{M_1}\langle 0|\Big)U_{C\bar{C}}^\dagger S_{CM_1}^\dagger\Big] \tag{A3}$$

$$= \mathrm{Tr}_{M_1}\Big[U_{M_1\bar{C}}\Big(|\Psi\rangle_{M_1\bar{C}}\langle\Psi|\otimes|0\rangle_C\langle 0|\Big)U_{M_1\bar{C}}^\dagger\Big] = \mathrm{Tr}_{M_1}\Big[U_{M_1\bar{C}}\Big(|\Psi\rangle_{M_1\bar{C}}\langle\Psi|\Big)U_{M_1\bar{C}}^\dagger\Big]\otimes|0\rangle_C\langle 0| \equiv \rho'_{\bar{C}}\otimes|0\rangle_C\langle 0|\,,$$

with $\rho'_{\bar{C}}$ as in Eq. (5).

APPENDIX B: DECOMPOSITION EQUATIONS

Here we give a decomposition of the state after applying the W operator of Eq. (2). This will allow us to estimate the fidelities for state transfer in terms of the relaxing properties of the map τ.

Let $|\psi\rangle_{C\bar{C}} \in \mathcal{H}_{C\bar{C}}$ be an arbitrary state. We notice that the C component of $W|\psi\rangle_{C\bar{C}}|0\rangle_M$ is always $|0\rangle_C$. Therefore we can decompose it as follows

$$W|\psi\rangle_{C\bar{C}}|0\rangle_M = |0\rangle_C\otimes\left[\sqrt{\eta}|0\rangle_{\bar{C}}|\phi\rangle_M + \sqrt{1-\eta}|\Delta\rangle_{\bar{C}M}\right] \tag{B1}$$

with $\eta\in[0,1]$ and with $|\Delta\rangle_{\bar{C}M}$ being a normalised vector of $\bar{C}$ and M which satisfies the identity

$${}_{\bar{C}}\langle 0|\Delta\rangle_{\bar{C}M} = 0\,. \tag{B2}$$

It is worth stressing that in the above expression η, $|\phi\rangle_M$ and $|\Delta\rangle_{\bar{C}M}$ are depending on $|\psi\rangle_{C\bar{C}}$. In a similar way we can decompose the vector obtained by acting with $W^\dagger$ on the first term of Eq. (B1), i.e.

$$W^\dagger|0\rangle_{C\bar{C}}|\phi\rangle_M = \sqrt{\tilde{\eta}}\,|\psi\rangle_{C\bar{C}}|0\rangle_M + \sqrt{1-\tilde{\eta}}\,|\tilde{\Delta}\rangle_{C\bar{C}M}, \tag{B3}$$

where $|\tilde{\Delta}\rangle_{C\bar{C}M}$ is the orthogonal complement of $|\psi\rangle_{C\bar{C}}|0\rangle_M$, i.e.

$${}_{\bar{C}C}\langle\psi|_M\langle 0|\tilde{\Delta}\rangle_{C\bar{C}M} = 0\,. \tag{B4}$$

Multiplying Eq. (B3) from the left with ${}_{C\bar{C}}\langle\psi|_M\langle 0|$ and using the conjugate of Eq. (B1) we find that $\eta=\tilde{\eta}$. An expression of η in terms of τ can be obtained by using Eq. (4). Therefore from Eq. (B1) and the orthogonality relation (B2) it follows that

$$\eta = {}_{\bar{C}}\langle 0|\tau^{L-1}\left(\rho'_{\bar{C}}\right)|0\rangle_{\bar{C}}, \tag{B5}$$

which, since τ is relaxing, shows that $\eta \to 1$ for $L \to \infty$. Moreover we can use [18] to claim that

$$\begin{aligned} 1-\eta &= |{}_{\bar{C}}\langle 0|\tau^{L-1}\left(\rho'_{\bar{C}}\right)|0\rangle_{\bar{C}} - 1| \\ &\leqslant \|\tau^{L-1}\left(\rho'_{\bar{C}}\right) - |0\rangle_{\bar{C}}\langle 0|\|_1 \\ &\leqslant K\,(L-1)^{d_{\bar{C}}}\,\kappa^{L-1}, \end{aligned} \quad \text{(B6)}$$

where $\|\Theta\|_1 = \sqrt{\mathrm{Tr}[\Theta^\dagger\Theta]}$ is the trace norm of the operator Θ, K is a constant which depends upon $d_{\bar{C}} \equiv \dim\mathcal{H}_{\bar{C}}$, and where $\kappa \in]0,1[$ is the second largest of the moduli of eigenvalues of τ.

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